Noah, Joseph, and Operational Hydrology

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Dedicated to Harold Edwin Hurst

. . . were all the fountains of the great deep broken up, and the windows of heaven were opened. And the rain was upon the earth forty days and forty nights. *Genesis, 6, 11-12*

. . . there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine . . . *Genesis, 41,* 29-30

Abstract. By 'Noah Effect' we designate the observation that extreme precipitation can be very extreme indeed, and by 'Joseph Effect' the finding that a long period of unusual (high or low) precipitation can be extremely long. Current models of statistical hydrology cannot account for either effect and must be superseded. As a replacement, 'self-similar' models appear very promising. They account particularly well for the remarkable empirical observations of Harold Edwin Hurst. The present paper introduces and summarizes a series of investigations on self-similar operational hydrology. (Key words: Statistics; synthesis; time series)

INTRODUCTION

By 'Noah Effect' we designate the fact that extreme precipitation can be very extreme indeed, and by 'Joseph Effect' the fact that a long period of high or low precipitation can be extremely long. In a series of papers to which the present work serves as Introduction and Summary, we shall describe in detail a family of statistical models of hydrology which we believe adequately account for the Noah and Joseph effects. Different papers in the series will be devoted, respectively, to mathematical considerations, to accounts of computer simulations, and to analyses of empirical records. Later papers will study various problems of water control engineering as problems of operations research, which they were long before the term 'operations research' itself was coined.

The models to be described were advanced and argued in *Mandelbrot* [1965] and *Mandelbrot and Van Ness* [1968]. We have carried out extensive experiments of every kind to test and develop these models, and we have, in our opinion, confirmed their soundness. It may be of interest to note that they are instances of a broad family of 'self-similar models.' The concept of 'self-similarity' originated in the theory of turbulence, to which it was long restricted, but it has recently become of value in studying a variety of natural phenomena (see for example, *Mandelbrot* [1963, 1966, 1967*a*, 1967*b*]).

A word of acknowledgment before we proceed. In investigations of current statistical models of hydrology, one of the most active groups has been that founded by Professor Harold A. Thomas, Jr., at Harvard. In view of the critical tone of much that follows, the authors hasten to express here their personal indebtedness to Harold Thomas. He directed B.B.M.'s curiosity to Hurst's work and to hydrology and later initiated J.R.W. into the intricacies of 'synthetic hydrology' and simulation.

PARTISAN COMMENTS ON CURRENT STATISTICAL HYDROLOGY

Current models of hydrology assume precipitation to be random and Gaussian (i.e., following the normal probability distribution, with its 'Galton ogive') with successive years' precipitations either mutually independent or with a short memory. 'Independence' implies in particular that a large precipitation in one year has no 'aftereffect' on the following years; 'short memory' means that all aftereffects die out within a few years. The classical short memory mechanism, the Gauss-Markov process, is a 'single lag linear autoregressive model.' In this case, aftereffects die out in geometric progression and decrease rapidly. More general are the 'multiple lag linear autoregressive models.' One feature common to all these models is their belonging to 'the Brownian domain of attraction,' a term which we shall define later. It is our basic belief that models in the Brownian domain cannot account for the Noah and Joseph effects. These models underestimate the complication of hydrological fluctuations and the difficulty of 'controlling them by establishing reserves to make the future less irregular' (to paraphrase the title of *Massé* [1946]).

Disappointment with *specific* models in the Brownian domain is today very widespread among hydrologists (for example, see *Yevjevich* [1968]). Therefore, our sweeping assertion can only be controversial in its blanket condemnation of all models in the Brownian domain. To try to minimize such controversy, we shall now describe various stop-gaps that have been proposed. We shall point out that, in effect, such models end up outside the Brownian domain.

Some authors eventually conclude that a description of hydrological reality requires a Gauss-Markov process with time-varying parameters. Such models must, however, be changed before their consequences have had time to develop fully. For example, before the sample average of precipitation has had time to 'stabilize' near its expected value, climatic change is assumed to modify that expectation. We believe such models to be rather pointless, because the usefulness of a statistical model lies primarily in its large sample predictions. Since a changing expected value easily overwhelms Gauss-Markov fluctuations, a Gauss-Markov hydrological model can be used only in conjunction with some 'master model' ruling climatic change. The over-all model, combining hydrology and climatology, is far from being of Gauss-Markov form.

Other approaches to hydrological modeling also start with a Gauss-Markov process and then introduce modifications that tend to be more extensive when records are long than when they are short. This general procedure can be illustrated using two examples. The first involves the loose but intuitive idea of the duration of a drought, the second the more rigorous but less intuitive concept of the Hurst range.

For droughts the point is that, if an independent Gauss process or a Gauss-Markov process is chosen to fit best the other aspects of precipitation, it will greatly underestimate the durations of the longest drought. Therefore, such processes must be modified by considering more durable aftereffects (for example, through 'multiple lag' models). One who considers such modifications as nuisance corrections to a basic Gauss-Markov process will naturally try to fit all available data with a 'minimal' modified process, having as short a span of aftereffects as possible. However, when the sample duration is sufficiently increased, 'unexpectedly' long droughts will again be observed. This shows, after the fact, that the 'minimal' model had attributed a special significance to the longest sample T that was available when it was constructed. As the sample increases, such a model must be changed. (For example, the number of lags must be increased.)

'Drought' being, as we said, an elusive concept, let us now proceed to the observed behavior of the 'Hurst range,' which is less intuitive but easier to study. To define it, one begins by evaluating the total capacity $R(s)$ a reservoir must have had, in order to perform 'ideally' for s years. 'Ideal performance' here means (a) that the outflow is uniform; (b) that the reservoir ends the period as full as it began; (c) that the dam never overflows; (d) that the capacity is the smallest compatible with (a), (b) and (c). The concept of an ideal dam is of course purely retrospective, since data necessary to design such a dam are only known when it is too late. However, the past dependence of the ideal capacity upon s tells a great deal about the long-run behavior of a river on which an actual dam is to be built. Postponing qualifications to later papers, let us describe a striking discovery H. E. Hurst made while examining $R(s)$ for the Nile and other geophysical records. Hurst divided the capacity $R(s)$ by the standard deviation $S(s)$ of s successive discharges. The empirical finding, then, is that *save perhaps for small values of s, the rescaled range $R(s)/S(s)$ is proportional to s^H with H a constant between 0.5 and 1.* Hurst judged H to be 'typically' near 0.7, but other estimates put H much higher, above 0.85. An independent Gauss model yields $R(s)/S(s) \sim 0.5$. Gauss-Markov models, 'multiple lag' models, and all other models in the Brownian domain give a more complex prediction: $R(s)/S(s) \sim s^{0.5}$ for large s, but $R(s)/S(s)$ grows faster than $s^{0.5}$ for small or moderate s, which we shall call the 'initial transient region.' In this transient region a variety of different behaviors may be obtained. Moreover, many models may lead to the same transient behavior, which makes them indistinguishable from the viewpoint of predictions concerning $R(s)/S(s)$. Then, if one has only the values of $R(s)/S(s)$ for $1 \leq s \leq T$ (with T a finite duration), many different models of the Brownian domain are likely to yield predictions undistinguishable from the data. When s exceeds T, however, the Hurst range of every one of these processes will soon merge into the classical $s^{0.5}$ pattern. So far such a convergence to $s^{0.5}$ has never been observed in hydrology. Thus, again, those who consider Hurst's effect to be a transient implicitly attach an undeserved importance to T, which is typically the currently available sample sizes. These scholars condemn themselves never to witness the full asymptotic development of the models they postulate.

TOWARDS A CHANGE OF DIRECTION IN HYDROLOGICAL MODELING

We have now sketched a few reasons, to be fully developed later, why we dislike hydrological models obtained by 'patching up' the Gauss-Markov process. It should be understood that this criticism does not imply that we expect to be able, with some other model involving few parameters, to represent fully the tremendously complicated hydrological reality. A model having few parameters can only be a 'first approximation,' and we believe that such a first approximation must endeavor to 'catch' the main features of the problem, namely the Joseph and/or Noah Effects.

To characterize our proposed non-Brownian first approximations, the loose distinction between 'low-frequency' and 'high-frequency' phenomena is useful. Using a Gauss-Markov process implies fitting high frequency effects first and worrying about low frequency effects later. We propose to invert this order of priorities. Conveniently, we shall be able to use the term 'low frequency' in either of its main meanings, to designate either a rarely occurring phenomenon, or an oscillating phenomenon with a long wavelength.

The concepts of 'low' and 'high' frequency are, of course, relative. Natural phenomena cover a continuous spectrum, in which very low frequencies of turbulence theory and very high frequencies of hydrology overlap around one cycle per day. This frequency, being fundamental in astronomy, may also separate zones in which intrinsically different mechanisms rule the fluctuations of precipitation. The same holds for the wavelength of one year. The third important wavelength in hydrology is 50 to 100 years, which we shall refer to as a 'lifetime.' This is roughly the horizon for which one designs water structures and also, coincidentally, the length of most hydrological records. The importance of this wavelength is of human, not astronomical, origin; it is purely 'anthropocentric.' Whereas precipitation fluctuations of wavelength near one day or one year may participate in several physical mechanisms, fluctuations of precipitation of wavelength near one lifetime are likely to participate in one mechanism only. Thus, the latter are likely to be simpler than the former. Now assume that one wants an approximation valid over a wide band of frequencies. It may be convenient to start by asking for a good fit in some narrow frequency band, with the hope that the formula so obtained will be applicable over the wide band. Under these circumstances, the band near one lifetime, although purely anthropocentric in its definition, constitutes in our opinion a better basis of extrapolation than the band near one year, which has meaning in astronomy.

We realize that a stress on low frequencies emphasizes idiosyncrasies. But the purpose of hydrological engineering is to guard against the recurrence of such idiosyncrasies, and one cannot afford to neglect any available information.

We also realize that, models in the Brownian domain having long been recognized as applicable in many fields of science (beginning of course with the Brownian motion of statistical mechanics), their proponents among hydrologists are often able to identify ready-made answers to the standard problems. Our proposed approach requires more work, but the answers appear to be sufficiently better to make this work worthwhile. Moreover, the concept of self-similarity, to be discussed later in the paper, will bring true simplicity.

'SMOOTH' AND 'ERRATIC' PROCESSES

We finally come to the promised characterization of the 'Brownian domain of attraction' and of related specific meanings for the terms 'smooth' and 'erratic' time series. We need three results of probability theory, two of which are classical, and all three of which relate to averages of T successive terms of a stationary time series designated by $X(t)$.

One says that $X(t)$ satisfies a law of large numbers when its average tends to a limit, the expectation $\mathcal{E}X(t)$, when $T \rightarrow \infty$. This law is the theoretical justification of the common practice of taking sample averages as estimates of population expectations.

One says that $X(t)$ satisfies the more demanding Central limit theorem in its original form when, for large T, the distribution of the average becomes approximately Gaussian, with a variance tending to zero as $T \rightarrow \infty$. This is the justification of the common belief that the sample average is likely, if T is not small, to be a good estimate of the expectation. A corollary of this is that, for large T, even the largest of the T quantities $T^{-1} X(t)$ contributes negligibly in relative value to the average $T^{-1} \Sigma_{t=1}^{T} X(t)$.

The final result is less well known but very important in applications. Let us call 'past average' the expression $T^{-1} \Sigma_{t=-T+1}^{0} X(t)$ and 'future average' the expression $T^{-1} \Sigma_{t=1}^{T} X(t)$ and consider the difference between these two. The third basic result on the averages of random sequences asserts that, as $T \rightarrow \infty$, the two averages can be considered independent, so that the variance of their difference is double the variance of each of them. It is unfortunate that this property does not yet have a generally accepted name. Let it be said immediately that this property fails to hold, either for fractional noises or for approximation thereto used in *Mandelbrot and Wallis* [1968*a*]. These processes satisfy, however, both the law of large numbers and the central limit theorem. If a natural phenomenon obeys the conditions of a validity of all three of these mathematical theorems, it will be called 'smooth' or 'in the Brownian domain of attraction.' There exist phenomena that fail to obey the conditions of validity of the third theorem, or of the last two, or even of all three. Such phenomena will be called 'erratic.' For example, the average $T^{-1} \Sigma_{t=1}^{T} X(t)$ may *fail* to tend to any limit. Or it may tend to a limit, whereas its distribution does not tend to the Gaussian. Or it may tend to a Gaussian limit, whereas 'past' and 'future' averages fail to become asymptotically independent. The importance of this latter circumstance for the hydrologist lies in the coincidental equality between the order of magnitude of most past records and the horizon of most designs (both equal one lifetime), and in the fact that, true expectations being unknown, planning requires the determination of the difference between the expected mean flow over a future lifetime and the known past average.

It is readily verified that the Gaussian models with a limited memory all assume hydrological phenomena to be 'smooth.' The Noah and Joseph effects, on the contrary, not only suggest that hydrological data are 'erratic' but also express the major two forms of erratic behavior. We shall speak of 'Joseph-erratic' behavior when the wettest decade within a century includes an extraordinary 'term' of wet years. We shall speak of 'Noah-erratic' behavior when a few of the years within the century witness 'floods' so major as to affect the average precipitation for periods of many years within which the flood years occurred. Needless to say, a process can be both Joseph- and Noah-erratic simultaneously, a complication that we shall face much later. 'Pure Joseph-erratic' behavior will be said to apply when none of the yearly precipitations during a 'wet term,' had it stood alone, would have been interpreted as a flood.

Clearly, the word 'erratic' should not be construed to suggest a 'black-and-white' contrast. The three theorems in question are, indeed, asymptotic, but scientific applications of mathematics always deal with T's in some finite horizon. Consider, for example, an infinite (nonrandom) series $a(t)$. For the mathematician, the basic distinction is whether the sum $\Sigma_{t=1}^{\infty} a(t)$ is, respectively, finite or infinite. For the scientist, on the other hand, the ultimate convergence of $\Sigma_{t=1}^{\infty} a(t)$ is of little import, unless $\Sigma_{t=1}^{T} a(t)$ is already close to its limit. Therefore, the concept of 'erratic' must be considered as coming in various degrees of intensity, rather like 'grey.'

THE ISSUE OF THE MARGINAL DISTRIBUTION OF THE YEARLY FLOW

We shall now characterize more accurately the idea of a 'pure Joseph-erratic' process, using the concept of 'marginal distribution,' which is defined as the distribution of the values of a process irrespective of their chronological order. We believe it reasonable to demand that, when the order of values of a pure Joseph-erratic sample is scrambled, one should be left with a smooth process. Thus, the marginal distribution of these values will draw a line between, on the one hand, Noah-erratic processes and, on the other hand, processes that are either smooth or pure Joseph-erratic.

The paragon of the pure Joseph-erratic is a process with a Gaussian marginal distribution. We should therefore, in every case, begin by checking whether the Gaussian marginal distribution applies. For this, in a first approximation, 'probability paper' plots are good enough. They evidently show that it is not exceptional that the marginal distribution be either nearly Gaussian or highly non-Gaussian. To stay near the land of Joseph, an example of nearly Gaussian marginal distribution is provided by the level of the Nile at the Rhoda Gauge near Cairo, an example of highly non-Gaussian by the annual discharge from Lake Albert. To find other examples of either behavior, it suffices to thumb through *Boulos* [1951]: Straight line interpolations are quite acceptable in certain cases, poor but perhaps bearable in some other cases, and dreadful in still others. Much less complete data are available in other places, but the existence of huge deviations from the Gaussian is very familiar: For example, runoffs due to major storms may appear on histograms as distant 'outliers.' Also, high water levels, which would be considered 'millenium floods' if one extrapolated the tails of the histogram from its body, occur much more frequently than they should under the Gaussian assumption.

Despite the importance of deviations from the Gaussian, we believe it worthwhile to begin our investigation of the Joseph Effect by Gaussian processes $X(t)$, which are by definition such that the joint distribution of their values at any finite number of instants is a multivariate Gaussian variable. Such processes will be examined in the next several sections. In the section near the end of the paper, highly non-Gaussian processes with a Noah Effect will be mentioned. (Processes that are only 'locally' Gaussian are studied in *Mandelbrot* [1968].)

GAUSSIAN PROCESSES AND THE COVARIANCE

Gaussian processes are well known to be fully specified by their covariance function: If $X(t)$ is of zero mean and unit variance, the covariance $C(s)$ is the correlation between $X(t)$ and $X(t + s)$. (Of course, in the case of Gaussian variables zero correlation is identical to independence.) Our problem is, then, to use the behavior of $C(s)$ to classify a Gaussian process as smooth or Joseph-erratic. What we need here is a distinction between a form of high-frequency effect—namely 'short lag' or 'short run' effects—and a form of low-frequency effect—namely 'long lag' or 'long run' effects. Short lag effects depend upon the values of $C(s)$ for a few small values of s. Long lag effects depend upon the other values of $C(s)$. We shall now examine this dichotomy on four examples. The first is the process of independent increments, whose covariance $C_1(s)$ satisfies $C_1(s) = 0$ for all $s \neq 0$. The second is the Gauss-Markov process of covariance $C_2(s) = \exp(-|s|/s)$. The last two are the processes of covariances respectively equal to $C_3(s) = (1 + |s|_2/s_3)^{-2}$ and $C_4(s) = (1 + |s|/s_4)^{-0.5}$, where s_2, s_3 and s_4 are constants.

The above four covariances differ considerably from each other in the long run, but $C_2(s)$, $C_3(s)$, and $C_4(s)$ are all three smooth and monotone in the short run. Therefore, if the sample duration is short, and the sample covariance correspondingly 'noisy,' the graphs of $C_1(s)$, $C_2(s)$, $C_3(s)$, and $C_4(s)$ may be undistinguishable, not only to the eye but even from the viewpoint of many tests of statistical significance that examine each value of s singly. That is, such statistical tests are liable to indicate that the differences between the sample covariance and either of the functions $C_1(s)$, $C_2(s)$, $C_3(s)$, and $C_4(s)$, are not statistically significant for most s. The statistician could then conclude that all the data will be acceptably fitted as soon as short run data have been fitted. Therefore, he will advise the hydrologist that there is no evidence that his data were not generated by an independent Gauss process (C_1), or a Markov-Gauss (C_2), or perhaps some more involved short-memory process.

This would, however, be a very rash conclusion. For example, under the hypothesis that the true covariance is $C_1(s)$, one would expect the relative proportion of positive to negative sample covariances to be about one. This proportion would be larger under the hypothesis $C_2(s)$, still larger under $C_3(s)$, and larger yet under $C_4(s)$. Thus, if statistical criteria geared towards low-frequency effects can be developed, it is reasonable to expect them to show the same data to be significantly closer to $C_3(s)$, or to $C_4(s)$, than to $C_2(s)$.

Our need, then, is to enhance long-run properties of a process while eliminating short-run wiggles. To do so, the best procedure is to integrate or to use moving averages (fancier averages will not be considered here). Three methods of dealing with long-run effects deserve to be singled out.

THE VARIANCE OF CUMULATED FLOWS

Let the 'accumulated flow' since time 1 be defined as $\Sigma_{u=1}^{t} X(u)$ and designated as $X^*(t)$. Then, G. I. Taylor's formula (see *Friedlander and Topper* [1961]) can be used to evaluate the variance of $\Delta X^*\ X^*(t+s) - X^*(t) = \Sigma_{u=t+1}^{s} X(u) = X(t+1) \cdots + X(t+s)$. This variance is given by $sC(0) + 2\ \Sigma_{u=0}^{s}\ (s-u)\ C(u)$, which immediately introduces a basic long-run dichotomy.

When $\Sigma_{u=0}^{\infty}\ C(u) < \infty$, var $[\Delta X^*] =$ var $[X^*(t+s) - X^*(t)]$ is found to be asymptotically proportional to $s\Sigma_{u=0}^{\infty}C(u)$, and $X(t)$ is found to be in the Brownian domain of attraction.

When, on the contrary, $\Sigma_{u=0}^{\infty}C(u)$ diverges sufficiently rapidly, var $[\Delta X^*]$ grows faster and X is not in the Brownian domain. When, for example, $C(u) = C_4(u)$, one finds that var $[\Delta X^*] \sim s^{1.5}$, where $\sim$ means 'asymptotically proportional to.' More generally, let $C(u) \sim u^{2H-2}$ for large u, with $0.5 < H < 1$. Then, for large s, var $[\Delta X^*] \sim s^{2H}$. Incidentally, assuming implicitly that $\Sigma_{u=0}^{\infty}C(u) < \infty$, G. I. Taylor suggested this infinite sum as a measure of the span of temporal dependence in a time series. The Taylor span is thus infinite for $C_4(s)$, finite and easy to estimate in cases like $C_1(s)$ or $C_2(s)$, where the series $\Sigma_{u=0}^{\infty}C(u)$ converges rapidly, and, finally, finite but hard to estimate in cases like $C_3(s)$, where the series $\Sigma_{u=0}^{\infty}C(u)$ converges slowly.

THE RANGE, THE JOSEPH EFFECT, AND HURST'S LAWS

Curiously, empirical data about the behavior of var $[\Delta X^*]$ in hydrology have been examined only after those relative to another measure of over-all behavior of a process, namely $R(s)/S(s)$, where the sequential range $R(s)$ was defined earlier to be the capacity of a reservoir capable of performing 'ideally' for s years and $S(s)$ to be the standard deviation of yearly flow. Among Gaussian processes, the dependence of $R(s)/S(s)$ upon s sharply distinguishes smooth from Joseph-erratic processes. This is already obvious for Joseph's own example of the seven years of high precipitation followed by seven years of drought, for which the ideal reservoir would have had to be enormous. If wet and dry years alternate at any point of a record, then ideal reservoir size is decreased. It is the task of mathematics to express this reduction in numbers.

First consider the case when $X(t)$ is an independent Gauss process. Then, when s is large, both $R(s)$ and $R(s)/S(s)$ equal $\sqrt{s}$, multiplied by some 'universal' random variable independent of s. The little that is known about those random variables is due to *Feller* [1951]. For the Gauss-Markov process and for other models for which the memory $\Sigma_{u=0}^{\infty}C(u)$ is finite, the '$\sqrt{s}$ law' remains true, but the multiplying random variables are changed.

The case of series exhibiting the Joseph Effect is entirely different. For such series, the $\sqrt{s}$ law *fails,* as was first noted by Harold Edwin *Hurst* [1951, 1956, 1965]. For hydrological series, as well as for many other natural time series, $R(s)/S(s)$ increases like Cs^H. Here, C and H are positive constants; H may range between 0 and 1 and is seldom near 0.5. We shall call this empirical finding 'Hurst's law.' Moreover, $\sqrt{\text{var}[\Delta X^*]}$, considered earlier, is also proportional to s^H rather than to $s^{0.5}$, as suggested by the usual simple models. This will be called 'Hurst's law for the standard deviation,' or 'Langbein's corollary of Hurst's law,' because it was first noted in Langbein's comments on *Hurst* [1956].

Strictly speaking, Hurst claimed a more demanding 'one parameter s^H law,' according to which $R(s)/S(s)$ is asymptotically equal to $(s/2)^H$. His reasons for claiming that $C = 2^{-H}$ are not too clear or convincing. Moreover, separate selection of H and of C can obviously ensure a better fit. It also yields a different estimate of H. For example, Ven Te Chow in his discussion of *Hurst* [1951] found a case where H goes up from 0.72 to 0.87 when C is separately estimated. Also, we found cases where the best estimate of H is below 0.5, which contradicts Hurst's assertion that $0.5 < H < 1$. See *Mandelbrot and Wallis* [1968c] for revised values of H.

Note also that Hurst's original '$(s/2)^H$ law' has proven dangerous. In some cases it tempted him, as well as other authors, to estimate H from a single sample of natural or simulated values of $X(t)$. Such estimates should be discarded. The revised statement we use means that estimation of H requires many values of s and, for every value of s, a large number of starting points t spread over the total sample of length T.

On the other hand, every specific model of the Joseph Effect, such as the fractional noise to be described in the sequel, will yield a relation between C and H, whose conformity with experience will test the value of the model.

SPECTRAL ANALYSIS: PRINCIPLE AND APPLICATION IN HYDROLOGY

In addition to the behavior of var (ΔX^*) and of Hurst's range, a third way of looking at low frequency phenomena is through spectral (or Fourier or harmonic) analysis. We only mention it here to say that the spectral density of hydrological records peaks sharply for very low frequencies, as is also the case for the so-called $l{:}f$ noises [*Mandelbrot,* 1967*a*]. A full discussion of this topic is postponed to the next paper.

RELATION BETWEEN THE JOSEPH EFFECT, HURST'S LAW, AND THE LONG RUN BEHAVIOR OF THE COVARIANCES

To account for the above listed findings concerning $\text{var}(\Delta X^*)$, $R(s)/S(s)$, or the spectrum has proved to be very hard. For example, perusal of the discussion of Hurst [1951] and *Hurst* [1951] demonstrates the kind of desperate expedient necessary to fit his finding within the universe of the simple statistical models. Claiming (incorrectly, as we shall demonstrate presently) that there exists no stationary random process with a range following the s^H law, several among the discussants have suggested either giving up statistical stationarity or invoking nonrandom 'climatic' changes.

A more hopeful reaction, already mentioned in the partisan comment at the beginning of this paper, is exemplified in such works as *Anis and Lloyd* [1953] and *Fiering* [1967]. These authors, and others, have constructed stationary stochastic processes of the usual kind (i.e., in the Brownian domain of attraction, satisfying $\Sigma_{u=0}^{\infty} C(u) < \infty$) for which both range and standard deviation are proportional to s^H over a finite span of values of s. But the usual $\sqrt{s}$ behavior still applies beyond this span. Thus, the s^H law is here a property of what we have called a transient span. This transient may be made arbitrarily long. But long transients can only be achieved with complicated processes with a long memory. For example, *Fiering* [1967] (p. 85) had to use an autoregressive model with *20 lags* to ensure that Hurst's law holds over the span $1 < s < 60$.

An alternative to this approach is based upon the existence of the self similar random processes, pointed out by *Mandelbrot* [1965] and examined below. For such processes, Hurst's law holds for all values of s. Even more important from our viewpoint, which emphasizes low frequency phenomena, is the existence of processes for which Hurst's law holds for the short as well as for the long run. For the standard deviation, this was already proved when we noted in passing that

$$\sqrt{\text{var}\,[\Delta X^*]} \sim s^H$$

whenever

$$C(s) \sim s^{2H-2}$$

The same asymptotic behavior can be shown to hold for the range $R(s)$ and the rescaled range $R(s)/S(s)$.

This observation is central to our study of the Joseph Effect. Before we examine it more closely, let us show how it can explain the existence of models in which Hurst's law holds in an initial transient. The key is that the values of var$[X^*(t + s) - X^*(t)]$ for $s < T$ are affected only by the values of $C(s)$ for $s < T$. Hence, changes in the covariance for $s > T$ will pass unnoticed when only the span $s < T$ is observable. Now suppose that, starting from (say) the covariance $C(s) = (1 + 10s)^{2H-2}$, long lag covariances ($s > T$) are decreased sufficiently to make $\Sigma_{u=0}^{\infty} C(u)$ convergent. The modified process $X(t)$ is thus 'brought back' into the Brownian domain of attraction. It could even be made into a 'multiple lag' autoregressive model, which is the usual generalization of the Markov model. For such modified processes, Hurst's law continues to hold for $s < T$, and for some time beyond $s = T$. On the long run, however, it will be replaced by the $\sqrt{s}$ law. For example, the standard deviation will equal $C\sqrt{s}$. The value of this constant depends upon the tail selected for the modified covariance. It is adjustable at will and quite arbitrary.

We consider such models, in which T plays a central role, to be *undesirable.*

DEFINITION OF SELF-SIMILAR ERRATIC GAUSSIAN PROCESSES

In criticizing the usual statistical models, as applied to hydrology, we don't underestimate their good features. In particular, wherever the independent Gauss process is an acceptable approximation, it is unbeatable. Where one must amend this process, there are features that one will want preserved. For example, all the models of the Brownian domain preserve the assumptions that the variance is finite, and Taylor's scale [defined as $\Sigma_{u=0}^{\infty} X(u)$] is also finite. But they destroy another property that makes the independent Gauss process so convenient to manipulate. We shall now express this property in an indirect way that will make it easier to generalize: We claim that the independent Gauss process $X(t)$ is so convenient because of the possibility of interpolating $\Sigma_{u=1}^{t} X(u)$ to continuous times with the help of a 'self-similar' random process $B(t)$, called Brownian motion (also called Bachelier process, or Wiener process). To define self-similarity, one must consider a portion of $B(t)$, with t varying from 0 to T, and rewrite it as $B(h\ T)$, with h varying from 0 to 1. 'Self-similarity' then expresses the fact that the rescaled function $T^{-0.5}B(h\ T)$ has the same distribution for every value of T.

From this, one immediately deduces that

$$\sqrt{\text{var}}\ [B(t + s) - B(t)] = s^{0.5} \quad \text{and}$$

$$\max_{0 \le u \le s} B(t + u) - \min_{0 \le u \le s} B(t + u) = Cs^{0.5},$$

with C a constant. These statements are forms of the $s^{0.5}$ law, but they are valid uniformly (that is, for all s) rather than asymptotically (that is, for high s).

By analogy, when studying the laws of Hurst, it is good to know that more general self-similar processes exist. A Gaussian process $X(t)$, of integral $B_H(t) = \int_0^t X(u)du$, is Joseph self-similar if the rescaled function $T^{-H}B_H(h\ T)$ is independent of T in distribution. That the s^H laws apply to $B_H(t)$ can be seen by simple inspection. $B_H(t)$ was called 'fractional Brownian motion' by *Mandelbrot and Van Ness* [1968].

Unfortunately, the derivative $B_H'(t)$, called 'fractional Gaussian noise,' is too irregular to be studied directly. As we interpolated the integral of the independent Gauss process by Brownian motion, we must now replace $X(t)$ by $B_H(t + 1) - B_H(t)$. The covariance of this function $B_H(t + 1) - B_H(t)$ is given for $s \geq$ by

$$C_H(s) = C[(s - 1)^{2H} + (s + 1)^{2H}]$$

with $0 < H < 1$. If $\frac{1}{2} < H < 1$ and s is large, one finds

$$C_H(s) \sim [2H(2H - 1)C]s^{2H-2}$$

This is precisely the form we have proposed to use to model phenomena obeying Hurst's law. Therefore, our models are approximations to fractional Gaussian noise.

APPLICATIONS OF SELF-SIMILARITY

To apply self-similarity, one proceeds very much as in the classical 'dimensional analysis' of fluid and solid mechanics. This is no accident. Hydrology can be considered the low frequency application of the theory of turbulence, in which self-similarity and dimensional analysis were introduced by von Karman and Kolmogoroff (*see Friedlander and Topper* [1961].

An illustration of dimensional analysis is encountered when dealing with flows of water in vessels of the same shape but of different dimensions and at proportionately different velocities. Then the calculations need not be repeated. It suffices to solve all relevant problems once. Solutions to other cases will be obtained by mere rescaling.

The application of self-similarity to hydrology is in the same spirit. Once one has performed the calculations relative to some 'reference' horizon, answers relative to other horizons can be obtained by simple rescaling. This may make it worthwhile, in the case of the reference horizon, to perform some very lengthy and involved calculations that would not otherwise be economic. The new hydrology we propose may demand readjustments of thought. But there is hope that the ultimate outcome of this new hydrology will be a set of new and better 'cookbook recipes.'

FORETASTE OF A DISCUSSION OF THE NOAH EFFECT

A discussion of the Noah Effect is several papers removed from the present introductory article. Our approach will resemble the methods *Mandelbrot* [1963] uses to describe the variation of commodity prices. The above considered function $X^*(t) = \Sigma_{u=1}{}^t X(u)$, with $X(u)$ the annual flow for the year u, will be the counterpart of the price of a commodity at the instant t. The very rapid and large changes typical of the behavior of prices will be compared to floods. Sporadic processes (see *Mandelbrot* [1967*c*]) will also be needed.

The Noah Effect certainly raises important problems in operational hydrology. Such problems are, however, separate from those raised by the Joseph Effect. In the next paper we shall show that the Noah Effect is not necessary to explain Hurst's findings, and in the third paper that it is insufficient.

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